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The wake region in the steady low-Reynolds-number flow past a cylinder

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Summary

Evidently the flow field defined in the title contains three rather than two discernable regimes. The first is the vicinity of the cylinder where the flow is viscosity dominated. The second is known as the outer field. There space variations are milder, viscosity is much less effective and the influence of the uniform parallel stream is dominant. In this regime the stream function representing the obstacle's disturbance is governed by the fourth-order Oseen equation. However, evidently its recorded solution fails over the half-plane downstream of the cylinder's axis. It is the author's contention that this failure reflects the existence of a third regime – the wake-where space variations are sharp but only in the transverse direction. To obtain a solution for the entire flow field an additional low-Reynolds-number expansion is constructed. It is matched with the well-known ones prevailing in the inner and outer regimes.

1. Introduction

In terms of cartesian co-ordinates (x, y), which are scaled with respect to the radius of the cylinder a, the governing equation for the flow problem under discussion reads

$$\mathrm{Re}\frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} = \nabla^4\psi. \tag{1}$$

Here ψ is the stream function which is normalized with respect to Ua^2. The undisturbed stream velocity is U and $+x$ directed. The kinematic viscosity is ν so that Reynolds number Re is equal to Ua/ν.

Inspired by various preceeding authors Proudman & Pearson [1] proposed the following form of low-Reynolds-number solution:

$$\psi = \psi^{(i)} \sim \Delta\psi_1^{(i)}(x, y), \tag{2}$$

$$\psi = \psi^{(0)} \sim \frac{1}{\mathrm{Re}} Y + \frac{\Delta}{\mathrm{Re}}\psi_1^{(0)}(X, Y). \tag{3}$$

Equations (2) and (3) express the solution in the inner and outer fields, respectively. In these relationships the gauge function Δ is given by

$$\Delta = (\ln(1/\mathrm{Re}) + k)^{-1}, \tag{4}$$

and Kaplun [2] suggested the following choice of k:

$$k = (\ln 4 + 1/2 - \gamma)$$

where γ is Euler's constant.

This form hinges on an assumption concerning characteristic lengths. Thus, in the inner field space variables are scaled with respect to a. Far from the obstacle the cartesian co-ordinates (X, Y) are scaled with respect to the viscous length (ν/U). The point made here is that a also characterizes transverse variations in a wake region which extends all the way to infinity. Therefore, in order to cover the field an additional, third, expansion $\psi^{(w)}(X, y)$ must be constructed and matched with $\psi^{(i)}$ and $\psi^{(0)}$.

2. Domain of validity of outer expansion

The wake region leaves an imprint on the recorded solution for $\psi_1^{(0)}$. It is barely perceptible and therefore has not been noticed for long. In fact it was accidently discovered when Bentwich and Miloh [3] developed a hitherto unknown integral form of expression for the disturbance stream function in the outer field. They have done that in their treatment of unsteady low-Reynolds-number flows past a cylinder. But for steady flows equation (24) of [3] can evidently be reduced to

$$\psi_1^{(0)} = -2\int_{-\infty}^{X} \frac{\partial}{\partial Y}(\ln(\tilde{R}) + \exp(\xi/2)K_0(\tilde{R}/2))\mathrm{d}\xi \tag{5}$$

where

$$\tilde{R}^2 = \xi^2 + Y^2$$

and K_0 is the modified Bessel function of the second kind. Changing variables one gets

$$\psi_1^{(0)} = 2\frac{\partial}{\partial Y}\int_0^{\infty}\left[\ln \hat{R} + \mathrm{e}^{\hat{X}/2}K_0(\hat{R}/2)\right]\mathrm{d}\eta \tag{5'}$$

where

$$\hat{X} = X - \eta, \quad \hat{R}^2 = \hat{X}^2 + Y^2.$$

This form is, in fact, the sum of two uniform distributions of harmonic and rotational singular solutions of the Oseen equation. The nature of the singularities of these two types of solutions at their local origin $X = \eta$, $Y = 0$ is similar but not identical. To be precise the small R expansion of $\mathrm{e}^{\hat{X}/2}K_0(\hat{R}/2)$ commences with $(-\ln \hat{R})$ which is cancelled by the first component of the integrand. However, the subsequent terms in that expansion, which are of the form $\hat{X} \ln \hat{R}$ and $\hat{R}^2 \ln \hat{R}$, remain uncancelled. Moreover, they are singular. Therefore one many not differentiate the integral of relationship (5′) with respect to Y across the horizontal half-plane downstream of the cylindrical obstacle's axis. It follows that Oseen's equation, which is a differential equation, is inapplicable there. Consequently, as contended and contrary to the prevailing assumptions, the outer field does not surround the cylindrical obstacle.

This incompleteness is, of course, also present in the unsteady solution developed in [3] which represents the flow caused by a cylinder departing impulsively from rest. In a subsequent paper by the same authors [4] it was completed by constructing an additional expansion which covers the region where the unsteady Oseen solution fails. That expansion represents the timewise evolution of the wake region. Of course, as time progresses the unsteady Oseen-flow solution of [3] approaches the steady one given by equation (5) or (5′) and the unsteady wake solution developed in [4] becomes that given in the next section. However, it is the author's view that one should study the classical problem of steady flow past a cylinder on its own, rather than as an eventual situation arrived at by the transient process considered in [3] and [4]. Therefore the incompleteness will be presently pointed to assuming that the flow is steady and has always been so.

Strictly speaking, in order to show that incompleteness, the arguments concerning the form (5) suffice. Furthermore, as a rule the onus is on whoever proposes a solution to define and prove its domain of validity. Therefore, it is those who claim that the Oseen solution can represent the field surrounding the obstacle that must substantiate this contention. However, under the circumstances the author has no choice other than lead an uphill fight and show that the recorded solution for $\psi_1^{(0)}$ and that represented by the form (5) or (5′) are one and the same and therefore fails to hold over the above-mentioned half-plane. Substitute in the following formula

$$\psi_1^{(0)}(X, Y) = \int_{(X'=-\infty)}^{(X, Y)} \left[\frac{\partial \psi_1^{(0)}}{\partial X'}(X', Y')\mathrm{d}X' + \frac{\partial \psi_1^{(0)}}{\partial Y'}(X', Y')\mathrm{d}Y' \right] \tag{6}$$

the well-known expressions for the steady disturbance velocity components as given in many texts like Van Dyke's [5],

$$\partial\psi_1^{(0)}/\partial Y = 2\{(\partial/\partial X)[\ln(R) + \exp(X/2)K_0(R/2)] - \exp(X/2)K_0(R/2)\}, \tag{7}$$

$$\partial\psi_1^{(0)}/\partial X = -2(\partial/\partial Y)[\ln(R) + \exp(X/2)K_0(R/2)]. \tag{8}$$

Then note that there are many ways to combine equations (6) (7) and (8) provided two conditions are satisfied. Firstly, the paths of integration must emanate far upstream where $\psi_1^{(0)}$ vanishes. Secondly, a pair of such paths leading to adjacent points must not form a curve enveloping the origin. The latter condition must be imposed in view of the singular nature of equations (7) and (8) and the multi-valuedness that may result thereof. It follows from the foregoing that the constant Y lines are legitimate paths of integration, whence the equivalence of the recorded solution for $\psi_1^{(0)}$ and the form (5).

The failure of the solution for $\psi_1^{(0)}$ over the half-plane downstream of the axis will now be explained. Clearly, equation (5) can be applied as it is, to calculate $\psi_1^{(0)}$ and any of its X- and Y-derivatives everywhere except over that plane. To obtain $\psi_1^{(0)}$ and its derivatives there, one must adopt the following procedure. First evaluate the appropriate integral along $Y = \pm\epsilon$ and then let ϵ approach zero. Such limit process yields

$$\psi_1^{(0)}(X, +0) = \psi_1^{(0)}(X, -0) = 0 \tag{9}$$

because the contributions of the Y-derivatives of the singularities of $\ln(\tilde{R})$ and $K_0(\tilde{R}/2)$

to the integral cancel out. However, this is not true for the third Y-derivatives of these. Thus fairly straightforward calculations produce the following result

$$-\frac{\partial^2 \psi_1^{(0)}}{\partial Y^2}(X, +0) = \frac{\partial^2 \psi_1^{(0)}}{\partial Y^2}(X, -0) = \pi \tfrac{9}{16}. \tag{10}$$

The first and third derivatives of $\psi_1^{(0)}$ which are even in Y, are evidently the same whether one approaches the positive X-axis from above or below. The double-valuedness implied by the result (10) renders the Oseen solution, and consequently also expansion (3), inapplicable at the horizontal half-plane under discussion. The limiting value of $\psi_1^{(0)}$ and its derivatives, which were obtained here, are used below in the construction of an additional expansion that holds over that plane and its immediate transverse vicinity.

Note that the result (10) is dissimilar to that obtained by simply differentiating equation (7) with respect to Y. Of course, this is precisely the point made herewith, namely that $\psi_1^{(0)}$ fails over the plane $Y = 0$, $0 < X < \infty$, just as it does at the origin. Relationship (7) and (8) cannot be differentiated there, and therefore both these regions should be excluded from the outer field. Admittedly the exclusion of the latter is much more obvious while the double-valuedness associated with the former was arrived at by a somewhat sophisticated integration process. However, both exclusions are direct consequences of the inapplicability of the Oseen model. Thus, by shrinking a finite-sized cylindrical obstacle to a line, as it is represented in the Oseen field, one produces infinite gradients there. The exclusion of that point is therefore obvious and universally accepted. But it has not been noticed that the troubles do not stop there. The Oseen equation accounts for vorticity convection, throughout the field, by the term $\partial(\nabla^2\psi)/\partial X$. Thus, according to that model, vorticity, which is generated at an enormous rate at the line obstacle, would have poured straight downstream, if indeed the downstream plane and its immediate transverse vicinity were included. It follows that the Oseen model cannot hold there.

It is of interest to note that the findings presented herewith hinge on physical and not only analytical premises. Indeed one could combine relationships (6), (7) and (8) subject to the above-said conditions and obtain a solution which is single-valued and continuous everywhere except, for example, $\theta = \pi/3$, $0 < R < \infty$. In view of the singular nature of relationships (7) and (8) this newly-defined solution fails over $\theta = \pi/3$, rather than $\theta = 0$. But such solutions are ruled out on physical grounds by invoking symmetry.

3. The wake region

The surface where the outer field fails is viewed here as a wake region. It is taken to be of a transverse width which is finite and of order a. Since this length is much smaller than ν/U, the wake region appears to be of zero thickness in the (X, Y) field. Such assumption is understandable from a physical view point. The region under discussion is likened to a 'shadow' cast by a cylinder placed in a parallel uniform stream.

It follows from that assumption that the transverse variations are characterized by a but not the axial ones. Therefore the co-ordinates associated with the wake are assumed to be (X, y). This is verified by successfully matching a solution developed for the wake region $\psi^{(w)}(X, y)$ with $\psi^{(0)}$ and $\psi^{(i)}$.

In fact assuming that $\psi^{(w)}(X, y)$ matches $\psi^{(0)}(X, Y)$ the form of expansion of the former is obtained by recasting $\psi^{(0)}(X, Y)$ in terms of (X, y) as follows

$$\psi^{(0)} = y + \frac{\Delta}{\mathrm{Re}}\left\{ \sum_{n=0}^{\infty} \frac{\partial^n \psi_1^{(0)}}{\partial Y^n}(X, \pm 0) y^n \,\mathrm{Re}^n \right\}, \quad y \gtrless 0. \tag{11}$$

In view of relationships (7) and (9) one gets

$$\psi^{(2)} \sim y + \Delta\{ y[2X^{-1} - \exp(X/2)K_1(X/2) - \exp(X/2)K_0(X/2)]\}, \tag{12}$$

and this is indeed the solution for the entire wake region. Within the assumed order of approximation it satisfies the appropriate equation because when recast in terms of (X, y) equation (1) reduces to

$$\frac{\partial^4 \psi^{(w)}}{\partial y^4} + \mathrm{Re}^2\left[2\frac{\partial^4 \psi^{(w)}}{\partial X^2 \partial y^2} + \frac{\partial(\partial^2 \psi^{(w)}/\partial y^2, \psi^{(w)})}{\partial(X, y)}\right] + O(\mathrm{Re}^4) = 0. \tag{1'}$$

It also matches $\psi^{(i)}$ when in that expansion only two terms are retained.

It has thus been demonstrated that a wake region exists and that its space variations in the axial and transverse directions are characterized by (ν/U) and a respectively. This has been done by calculating the leading terms in expansions (2), (3) and (12). It is also noted that matching with additional terms in expansions (3) and (2), which are known to be of the $O(\Delta^n/\mathrm{Re})$ and $O(\Delta^n)$, would persist by adding terms of $O(\Delta^n)$ to expansion (12).

It is finally observed that subsequent terms in the summation of equation (11) are of $O(\mathrm{Re}^j\,\Delta)$, $j = 1, 2$. These are negligible compared with Δ^n for any n. Consequently in the context of the construction of an asymptotic expansion for $\psi^{(w)}$ the double-valuedness implied by the result (10) is of no consequence.

4. Discussion

There is ample evidence that wakes trail obstacles placed in uniform streams when Reynolds number is high. Their presence in low-Reynolds-number flows is hardly observable experimentally and analyses recorded to date account for such region but only implicitly. This work is an explicit exposure of the wake in such flow.

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